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Technical Note

On estimating thermal diffusivity using analytical inverse solution for unsteady one-dimensional heat conduction

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Abstract

A modified procedure for calculating the thermal diffusivity of solids based on temperature measurements at two points and the semi-infinite boundary condition is presented. The method makes use of a solution to the unsteady one-dimensional inverse heat conduction problem for the semi-infinite solid. The procedure gives accurate results based on temperature changes produced by an arbitrary fluctuating heat flux source at the boundary.
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1. Introduction

Many creative and innovative methods are available for determining thermal diffusivity of solids. These range from Kelvin's approach to determine the effective thermal diffusivity of soil based on 24 h periodic application of sunlight [1] to modern sophisticated methods using laser pulses and infrared detectors to measure the time for the peak in a thermal wave to pass through a thin sample [2]. For numerous applications of practical interest, even a rudimentary heating device and a set of temperature sensors may be sufficient to obtain useful thermal property data. Recently, Monde and Mitsutake [3] proposed a method for determining the thermal diffusivity of solids using an analytical inverse solution for unsteady heat conduction based on measurements at two points within the solid and the semi-infinite boundary condition. The merit of their method was independence of the solution to the boundary heat flux used to produce the temperature change in the solid. They demonstrated accurate predictions of thermal diffusivity for quite different surface boundary conditions provided temperature changes occurred reasonably smoothly. In this communication we show that their procedure can become robust to a more arbitrarily fluctuating boundary condition if the techniques for improving practical application of the inverse solution proposed by Woodfield et al. [4] are incorporated in the method. This has the advantage that even a poorly controlled external boundary condition, could be used to determine accurately the diffusivity of a thermally thick solid with two sensors inserted in the interior.

2. Formulation

Eqs. (1a)–(1d) define the problem of interest. The formulation is very similar to that in [3] except that Eq. (1d) contains the additional correction terms ($j = 1, N_{corr}$) proposed in [4]. Eq. (1d) is the point where the thermocouple measurements enter the problem. Notice that the sensor

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Nomenclature

a	thermal diffusivity	s	laplace transform variable
C_1	coefficient relating to size of data correction windows	t	time
C_2	Fourier number relating to how much future data included in curve fit	T	temperature
$F(a)$	function to be minimized to obtain a	$W_{j,k}$	coefficient defined by Eq. (9)
$G_{j,k}$	coefficient defined by Eq. (5)	z	depth from the surface
L	thickness of sample	z_1	position of first sensor
$P_{j,k}$	coefficient determined by least-squares method for data fit	z_2	position of second sensor

is at a position $z = z_n$ and no reference is made to the boundary condition at $z = 0$. The basic concept is to solve Eq. (1) for two different sets of measurements, one taken at $z = z_1$ and the other at $z = z_2$. The thermal diffusivity, a, is then chosen by a least-squares method so that the two answers agree over the time range of the measured data. As pointed out in [4], Eq. (1c) is valid to better than 1% accuracy provided $at/L^2 < 0.1$ where L is the thickness of the sample.

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial z^2} \tag{1a}$$

$$T|_{t=t_0} = T_0 \tag{1b}$$

$$T|_{z\to\infty} = T_0 \tag{1c}$$

$$(T - T_0)|_{z=z_n} = \sum_{j=0}^{N_{\text{corr}}}\sum_{k=0}^{N_k} P_{j,k}^{(n)} \frac{(t-t_j)^{k/2}}{\Gamma((k/2)+1)} \tag{1d}$$

It is important that Eq. (1d) accurately approximates the measurements. As in [4], the coefficients $P_{j,k}$ for Eq. (1d) are determined by the linear least squares method over the whole data range for $j = 0$ and then corrections to the fit are made over successively smaller windows of data for $j = 1, N_{\text{corr}}$. The range of data (t_j to t_f) corresponding to the jth correction term in Eq. (1d) is determined according to Eqs. (2) and (3) where $C_1 = 0.7$, $C_2 = 0.8$ and t is the time at which the surface temperature is evaluated.

$$t_j = t_f - C_1(t_f - t_{j-1}) \tag{2}$$

$$a(t_f - t)/z_1^2 = C_2 \tag{3}$$

Eq. (1) can be solved at any z position. For the inverse case where $z < z_n$, the solution proposed by Monde [5] in the form presented in [4] is given by Eq. (4a). If $z > z_n$ the problem changes from an IHCP to a direct conduction problem and the solution is given by Eq. (4b).

$$T_n(z) - T_0 = \sum_{j=0}^{N_{\text{corr}}}\sum_{k=-1}^{N_k} \frac{G_{j,k}^{(n)}(t-t_j)^{k/2}}{\Gamma((k/2)+1)} \quad (z < z_n) \tag{4a}$$

$$T_n(z) - T_0 = \sum_{j=0}^{N_{\text{corr}}}\sum_{k=0}^{N_k} P_{j,k}\left(4(t-t_j)\right)^{k/2} i^k \text{erfc}\left((z-z_n)/\left(2\sqrt{a(t-t_j)}\right)\right) \quad (z \geqslant z_n) \tag{4b}$$

In Eq. (4a) $G_{j,k}^{(n)}$ is a function of z which satisfies the series given by Eq. (5) when coefficients of like powers of s are equated.

$$\sum_{k=-\infty}^{N_k} \frac{G_{j,k}^{(n)}}{s^{(k/2)+1}} = \sum_{m=0}^{N_k} \frac{P_{j,m}^{(n)}}{s^{(m/2)+1}} \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{z_n - z}{\sqrt{a}}\right)^i s^{i/2} \tag{5}$$

If we have measurements at two points in the solid, $z = z_1$ and $z = z_2$ then we wish to find the thermal diffusivity, a, such that it minimizes Eq. (6) for a specified surface, z.

$$F(a) = \sum_{i=1}^{N_{\text{time}}} (T_1(z) - T_2(z))^2 \tag{6}$$

$T_1(z)$ is calculated from Eq. (4) with $n = 1$ and $T_2(z)$ with $n = 2$. The sensitivity of the estimate of a to the position of z will be discussed below. N_{time} is the number of discrete points in time where the solution is evaluated. Note that in reference [3] the function $F(a)$ was made an integral of time rather than the summation over several points in time as in Eq. (6). The summation is more appropriate for the present formulation since for each point in time the solution is evaluated independently [4].

Differentiating Eq. (6) with respect to diffusivity, a, we obtain Eq. (7) for the minimum.

$$\frac{\partial F(a)}{\partial a} = \sum_{i=1}^{N_{\text{time}}} 2(T_1(z) - T_2(z))\left(\frac{\partial T_1}{\partial a} - \frac{\partial T_2}{\partial a}\right) = 0 \tag{7}$$

Differentiating Eq. (4) with respect to a, we obtain Eq. (8).

$$\frac{\partial T_n}{\partial a} = \sum_{j=0}^{N_{\text{corr}}}\sum_{k=-1}^{N_k} \frac{W_{j,k}^{(n)}(t-t_j)^{k/2}}{\Gamma((k/2)+1)} \quad (z < z_n) \tag{8a}$$

$$\frac{\partial T_n}{\partial a} = \sum_{j=0}^{N_{\text{corr}}} P_{j,0}\frac{z_n - z}{2a^{3/2}}\left(4(t-t_j)\right)^{-1/2}\left(\frac{2}{\sqrt{\pi}}\right)\exp\left(-(z-z_n)^2/(4a(t-t_j))\right) + \sum_{j=0}^{N_{\text{corr}}}\sum_{k=1}^{N_k} P_{j,k}\frac{z_n - z}{2a^{3/2}}\left(4(t-t_j)\right)^{(k-1)/2} i^{k-1}\text{erfc}\left((z-z_n)/\left(2\sqrt{a(t-t_j)}\right)\right) \quad (z \geqslant z_n) \tag{8b}$$

In Eq. (8a) the coefficients, $W_{j,k}$ ($k=-1,N_k$) are obtained by equating coefficients of like powers of s in Eq. (9).

$$\sum_{k=-\infty}^{N_k} \frac{W_{j,k}^{(n)}}{s^{(k/2)+1}} = \sum_{m=0}^{N_k} \frac{P_{j,m}^{(n)}}{s^{(m/2)+1}} \sum_{i=1}^{\infty} \frac{-i}{i!2a}\left(\frac{z_n - z}{\sqrt{a}}\right)^i s^{i/2} \tag{9}$$

Making use of Eq. (8), Eq. (7) can be solved quite easily numerically to obtain the diffusivity, a, in a similar manner to [3] by the bisection method or some other search method. For most cases, there tends to be only one positive root for Eq. (7). Also generally, a negative value of $dF(a)/da$ means the present estimate of a is too low and a positive value means a is too high. It should be noted however, that if the data contains strong positive and negative fluctuations it might be possible for Eq. (7) to have more than one root. Therefore it is important to confirm that the final estimated diffusivity does bring about a good match by calculating the temperature at z_2 based on the readings at z_1 and the semi-infinite boundary. This can be done using Eq. (4b). Such numerical problems are less likely to arise if temperatures either increase or decrease monotonically.

3. Estimating other thermal properties

Noting Eqs. (4a) and (4b), one may wonder what the advantage is of including the inverse solution, Eq. (4a), since the direct solution alone should be sufficient. The inverse solution has the merit that it also gives us the heat flux divided by thermal conductivity at $z = 0$. Therefore, if in an experiment, the heat flux is measured, for example by measuring power to an electrical heater, this extra information can be used to determine the thermal conductivity in addition to the diffusivity [3]. Moreover, if the density is known, then the definition of thermal diffusivity will give us the specific heat capacity.

4. Results

We consider some computer-generated data to establish the mathematical accuracy of the above procedure. To make the example of practical interest, we select an insulated carbon steel rod $L = 50$ mm exposed at one end to different heat flux boundary conditions. Sensors are positioned at 2 mm and 5 mm from the surface. Data is generated using analytical solutions given in [1]. A random fluctuation taken from the Gaussian Normal distribution ($1.96\sigma = 0.1$ K) is added to each reading to simulate the effect of noise in the data. Three different heat flux boundary conditions are considered, a constant heat flux, a sinusoidal heat flux and a square wave. To produce a reasonable temperature gradient in the solid during the simulation the peak heat flux in each case is set at 100 or 200 kW/m^2. The time period, τ, of the oscillating cases is greater than 3 s ($a\tau/z_2^2 > 1.1$), which is more than large enough to ensure the oscillations are detected at both depths.

Fig. 1 shows the effect of changing z in Eq. (7). If $z \geqslant z_2$ then the formulation is a direct heat conduction problem and Eq. (4b) is used for both depths. If however, $z > z_1$ the inverse heat conduction result is used. Of the three cases in Fig. 1, the constant heat flux result is the most accurate and the square-wave case shows the greatest departure from the true value for thermal diffusivity.

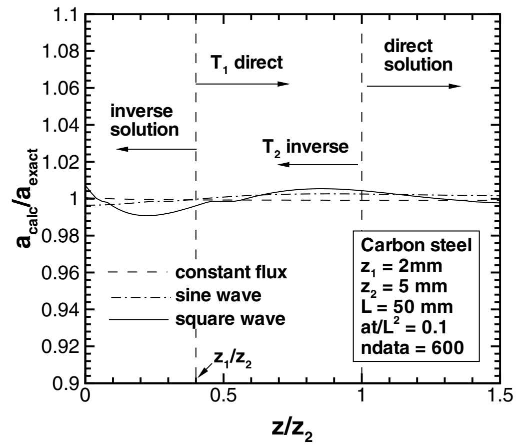

Fig. 1. Effect of location of calculation plane on estimate of thermal diffusivity.

Regardless of z position, all results are within plus or minus 1% of the exact result. Thus the procedure is not greatly sensitive to type of boundary or to the choice of the calculation plane position. Having said this, it is not desirable that z be too much larger than z_2 since the calculated result tends to vanish with depth into the solid. Possibly either $z = z_1$ or $z = z_2$ may be good choices since the measured values for $T_1(z_1)$ or $T_2(z_2)$ can be substituted directly into Eq. (7) reducing the required calculation effort. As mentioned above, $z = 0$ is a useful choice for the calculation plane position when the heat flux is known and one also wishes to determine the thermal conductivity.

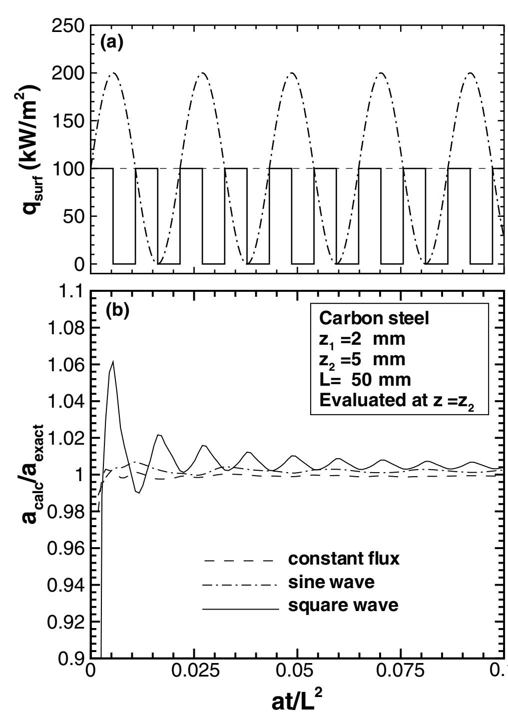

Fig. 2. Effect of time range on calculated thermal diffusivity. (a) boundary condition at surface and (b) calculated thermal diffusivity.

All results in Fig. 1 were calculated using 600 data points from $at/L^2 = 0$ to $at/L^2 = 0.1$. Fig. 2 shows the effect of changing the time range used to calculate a. For each point in Fig. 2(b), data are used from 0 to at/L^2 to calculate a. In Fig. 2(a) the boundary condition at the surface used to produce the temperature change is shown. Note that the present method is not modified in any way to suit the different heat flux boundary conditions, but simply the coefficients in Eq. (1d) change to fit the temperature data.

Fig. 2 shows that the accuracy of the method diminishes if a few data points over a short time ($at/L^2 < 0.01$) are used to calculate the diffusivity. As in Fig. 1, the constant heat flux case is better than the oscillating cases. When at/L^2 approaches 0.1 the accuracy of all three cases falls within the range of plus or minus 1%. Therefore, consistent with [3] the present results demonstrate that it is better to use close to the full time range for which the solid may be considered semi-infinite to estimate the thermal diffusivity.

5. Conclusion

By incorporating the procedures suggested by Woodfield et al. [4] into the method by Monde and Mitsutake [3], it is possible to calculate accurately the thermal diffusivity for an arbitrary fluctuating boundary condition. The procedure was demonstrated using computer-generated data for a surface heat flux that varies as a sine wave or a square wave in time. Both direct and inverse solutions gave a similar accuracy for the cases considered.

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